Reassesment of the Skin Depth Effect

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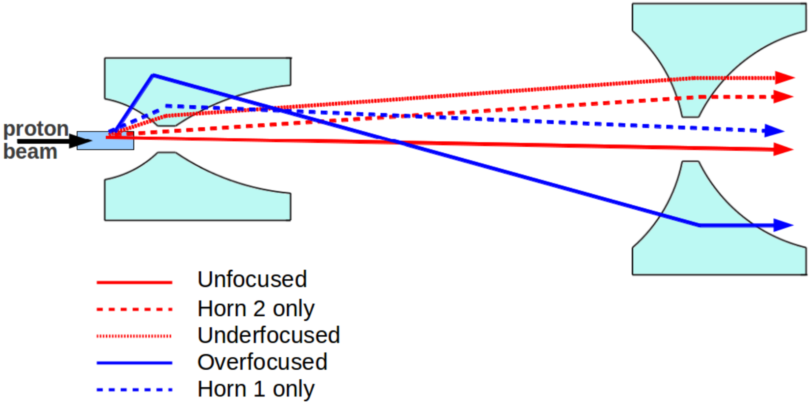
March 14, 2014

Outline

Horn Focusing

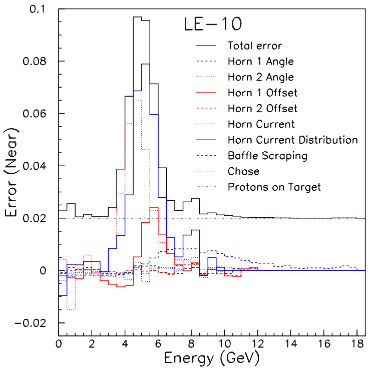
Horn focusing, and hence neutrino flux, affected by the distribution of current (actually magnetic field) inside of the inner conductor.



Largest effect on particles which pass through the conductor at the neck.

Significant Effect?

Since early on in MINOS we have believed this was a significant effect.

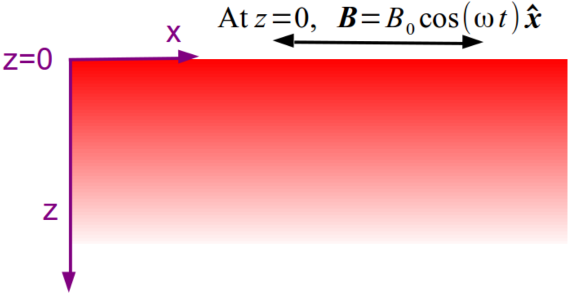


Recent LBNE work showed a small effect. What gives?

Outline

Semi-infinite plane

First, a simple example. Consider a semi-infinite conducting plane with an oscillating magnetic field at the surface.



Is there a field in the bulk? If so, what is its distribution?

The diffusion equation

Following Jackson 5.18, as long as the wavelength $\lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$ is much larger than the size of the system we are in the "quasi-magnetostatic" regime. In that case **J** obeys the diffusion equation and, more usefully, so does **H**:

$$\nabla^2 \mathbf{H} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t}$$

For us $\tau = \frac{1}{f} = 4.6ms$ so $\lambda \approx 10^6 m$ and we are safely in the quasi-static regime. Note: I am using S.I. units.

Solving the semi-infinite plane

Fact

By symmetry, H can only be a function of z and t.

Method

We'll solve for $H = H_x(z, t)$ and posit a separable solution of the form $H_x(z, t) = Z(z)T(t)$:

$$\frac{\partial^2}{\partial z^2}(ZT) = \mu\sigma \frac{\partial}{\partial t}(ZT)$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \mu\sigma \frac{1}{T}\frac{\partial T}{\partial t}$$

Solving the semi-infinite plane

Fact

Both sides of this equation are constants:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \mu\sigma\frac{1}{T}\frac{\partial T}{\partial t} \equiv -k^2$$

This is because the LHS is only a function of z and the RHS is only a function of t, and the two are independent variables.

Solution for $T(t)$

$$-k^2 = \mu\sigma\frac{1}{T}\frac{\partial T}{\partial t}$$

It's easiest to allow the field to have a complex time dependence and then find the physical fields at the end by taking the real part of the solution.

Take $T(t) = H_0 e^{-i\omega t}$ so $k^2 = i\omega\mu\sigma$.

Solving the semi-infinite plane

Solution for $Z(z)$

Now, we have to solve:

$$\frac{\partial^2 Z}{\partial z^2} + k^2 Z = 0$$

Since it's a second order equation there are two independent solutions:

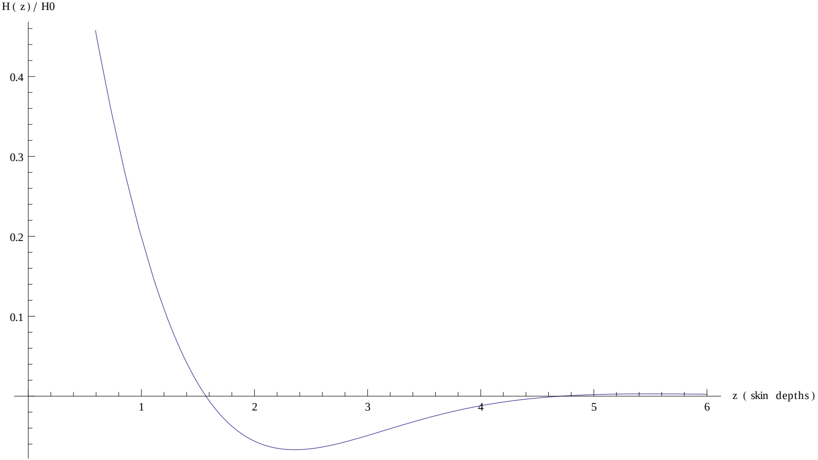
$$Z(z) = Ae^{kz} + Be^{-kz}$$

Solving the semi-infinite plane

Boundary Conditions

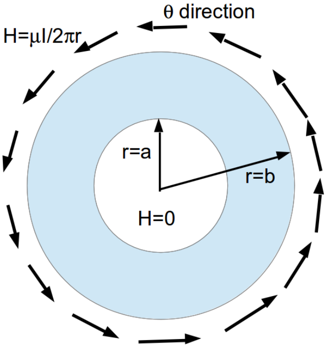
- As $z \to \infty$ the field must not blow up. Thus, assume $\Re[k] > 0$ and then we must have $A = 0$.
- At $z = 0$ the magnitude of the field must be H_0. Thus $B = 1$.
- So $H(z,t) = \Re\left[H_0 e^{-kz} e^{-i\omega t}\right]$.
- Here $k = (1+i)\sqrt{\frac{\omega\mu\sigma}{2}}$ and $\delta \equiv \sqrt{\frac{2}{\omega\mu\sigma}}$ is the "skin depth".
- The final solution is $H(z,t) = H_0 e^{-z/\delta} \cos(z/\delta + \omega t)$

Solving the semi-infinite plane



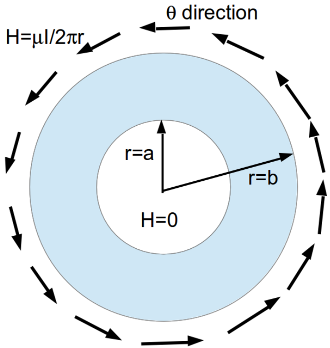
A solution for the horns

The horns have the geometry of a cylindrical pipe, with the field along $\hat{\theta}$, the radial coordinate is r, and z is the distance along the beam axis.



A solution for the horns

The the inner conductor begins at $r = a$ and ends at $r = b$. These radii are functions of z.



A solution for the horns

Method

- We again have to solve the diffusion equation, but this time in cylindrical coordinates.
- We know that $H \neq H(\theta)$ due to the symmetry.
- The field is zero at $r = a$ and at $r = b$ we have $H(b) = \mu I/(2\pi b)$.
- We'll solve for fixed z, incorporating the z dependence via the boundary conditions. I believe this is OK, but not 100% sure.

A solution for the horns

Method

- The time dependence is the same as for the semi-infinite plane.
- In cylindrical coordinates the radial portion of the diffusion equation reads:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 R = 0$$

This is a Bessel equation.

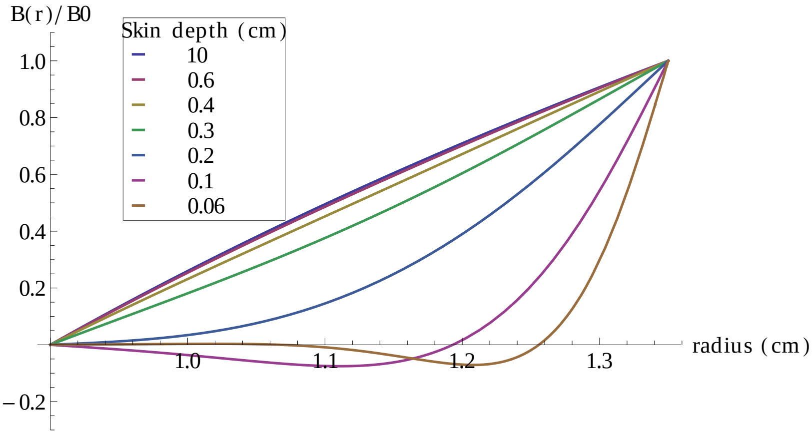
A solution for the horns

Method

- $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 R = 0$
- It has solutions $J_0(kr)$ (Bessel function of the first kind) and $N_0(kr)$ (Bessel function of the second kind – a.k.a. Neumann function).
- $R(r) = AJ_0(kr) + BN_0(kr)$ and then solve for A and B in terms of the known values of the field at a and b.

The magnetic field in the inner conductor

The field is normalized to the field at $r = b$.



Obtaining the current

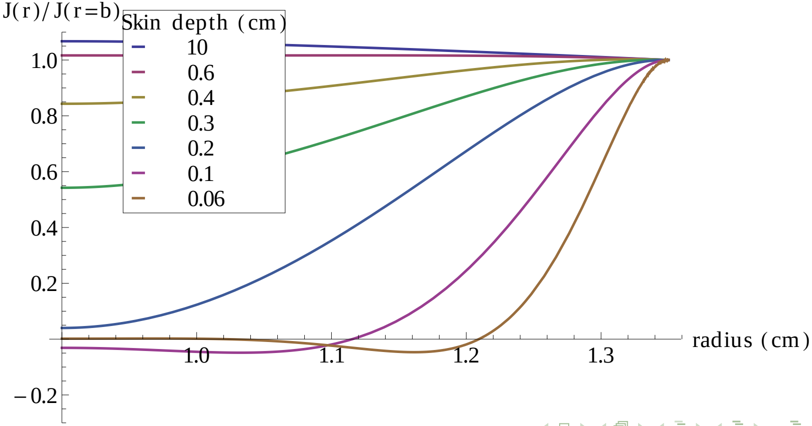
- The current is calculated from the field using $\nabla \times \mathbf{H} = \mathbf{J}$
- Since $\mathbf{H} = H\hat{\theta}$, and there is no theta dependence, the only non-zero term is:

  $\mathbf{J} = -\frac{\partial H}{\partial z}\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rH)\,\hat{z}$
- We need derivatives of Bessel functions: $\frac{\partial}{\partial r}J_0(kr) = -kJ_1(kr)$ and likewise for N_0.
- Considering only the $\hat{z}$ component, we get:

  $J_z = -\frac{H}{r} - \frac{\partial H}{\partial r} = -\frac{H}{r} + k\left[AJ_1(kr) + BN_1(kr)\right]$
- The $\hat{r}$ component is interesting but messy to calculate since one needs the analytic shape of the horns. More later.

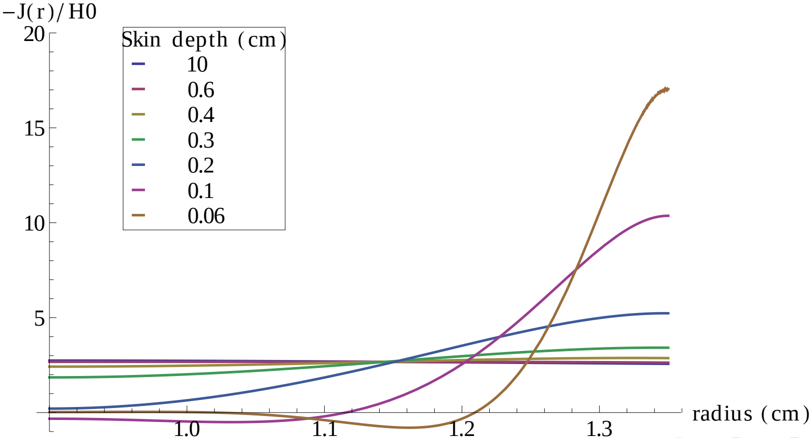
The current in the inner conductor

Normalized as $J(r)/J(b)$



The current in the inner conductor

Normalized as $-J(r)/H_0$



Outline

MINOS-doc-1283

- The old reference is MINOS-doc-1283 *Systematic Uncertainties in the NuMI Beam Flux*.
- The document discusses the semi-infinite plane case.
- It also discusses the cylindrical case, but a solution of the form $J(r) = AJ_0(kr) = A\left[\mathrm{ber}(\sqrt{2}r/\delta) - i\ \mathrm{bei}(\sqrt{2}r/\delta)\right]$ is assumed.
- Note, the solution is for the current, rather than the magnetic field.
- The value of A is found by normalizing the magnitude of the current density to the density at the outer edge:

$$\left|\frac{J(r)}{J(b)}\right| = \sqrt{\frac{\mathrm{ber}(\sqrt{2}r/\delta)^2+\mathrm{bei}(\sqrt{2}r/\delta)^2}{\mathrm{ber}(\sqrt{2}b/\delta)^2+\mathrm{bei}(\sqrt{2}b/\delta)^2}}$$

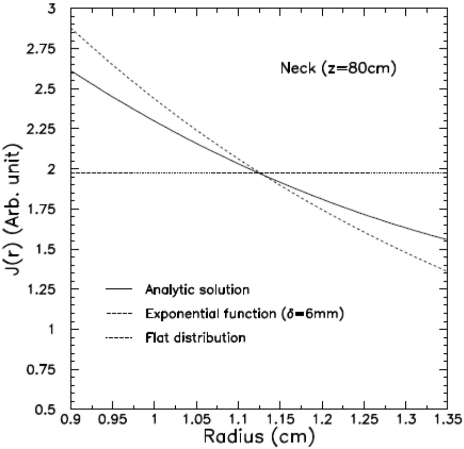
- Then, the integral of the current density is made to add up to the actual current in the horns (185 kA).

MINOS-doc-1283

What's wrong with this?

- The solution is justified by citing the textbook by Marion, but the textbook is deriving the current in a current carrying wire!
- In that case $B = 0$ because $N_0(kr)$ blows up as $r \to 0$. But, that's not a relevant boundary condition for the horns.
- And hey, shouldn't we be finding the real portion of the current density, rather than computing the magnitude of the complex density?? These are not the same thing!
- Also, the factor of $\sqrt{2}$ doesn't look right to me given that the document states a calculated skindepth $\delta = 7.7\,\mathrm{mm}$ which I understand to be correct. But, when I compute that, the $\sqrt{2}$ is inside δ.
- Finally, even if you follow MINOS's prescription you can't reproduce the plots in MINOS-doc-1283.

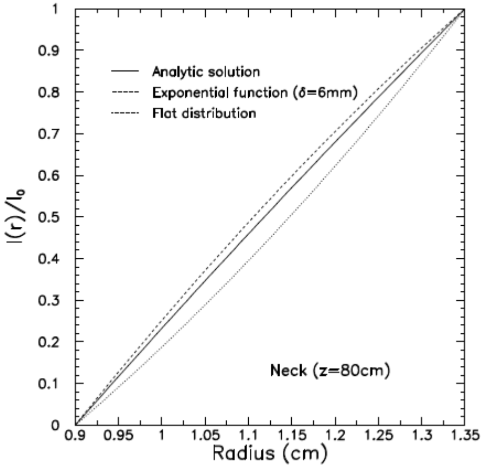
Current distribution from MINOS-doc-1283



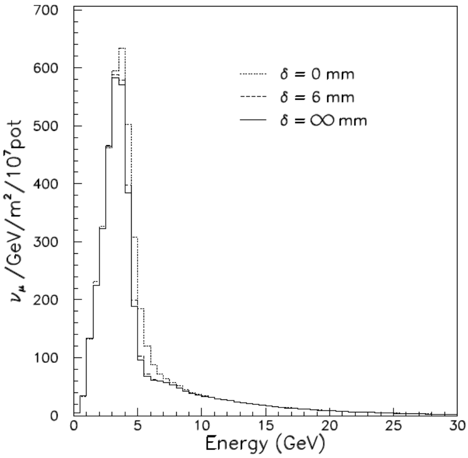
Considering the semi-infinite plane derivation, the current is at the wrong edge of the conductor!

Integrated current from MINOS-doc-1283

Note, this is the current enclosed within the radius r. The lower curve is the flat distrubution.



LE10 spectrum from MINOS-doc-1283

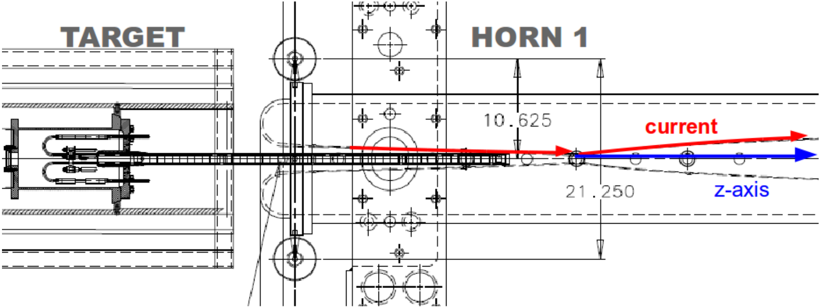


The effect goes in the wrong direction!

Outline

The radial current

A radial component of the current is predicted from Maxwell's equations and shouldn't be too surprising:

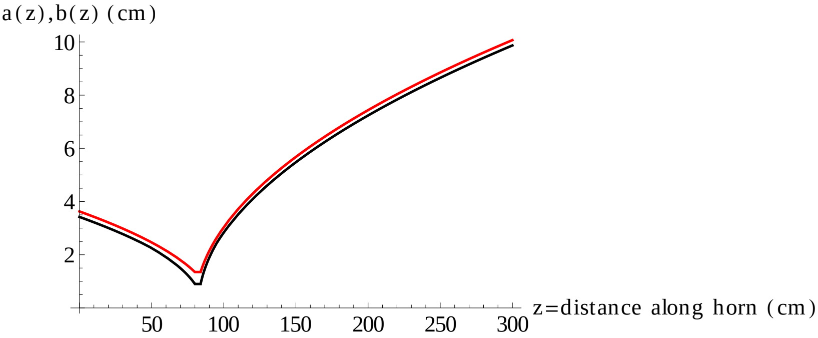


A calculation

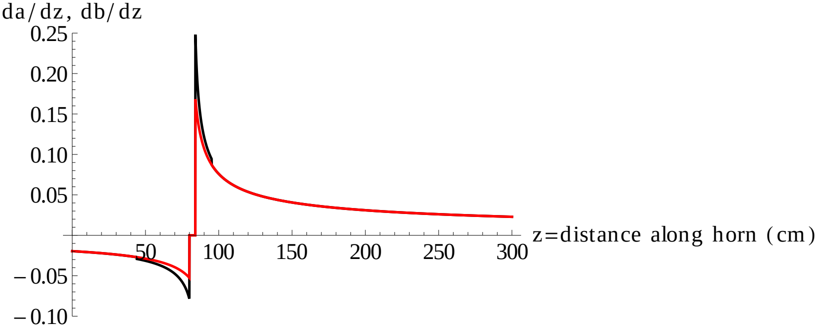
- Recall $H \equiv H_\theta = A(a,b,k)J_0(kr) + B(a,b,k)N_0(kr)$ for fixed z.
- We want $J_r = -\frac{\partial H}{\partial z}$
- The z dependence enters the problem via the radial location of the boundaries: $a(z)$ and $b(z)$
- Then, we compute $\frac{\partial H}{\partial z} = \frac{\partial H}{\partial a}\frac{\partial a}{\partial z} + \frac{\partial H}{\partial b}\frac{\partial b}{\partial z}$
- In the simple case that the horn is of uniform thickness ΔT then $b = a + \Delta T$ and $\frac{\partial H}{\partial b} = 0$. So, $J_r = -\frac{\partial H}{\partial a}\frac{\partial a}{\partial z}$.
- Note that $\frac{\partial a}{\partial z}$=arctan($\theta$) is tangential to the horn conductor surface, in the direction of the current we drew earlier.

The inner conductor - $a(z)$ and $b(z)$

My best estimate of the shape from a piecewise function in a blurry old document:

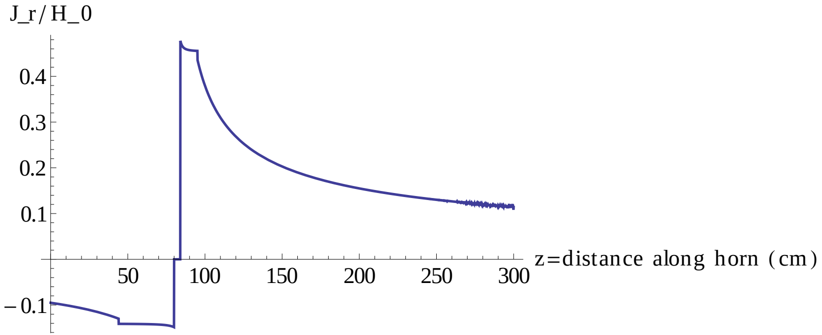


The inner conductor - $\partial a/\partial z$ and $\partial b/\partial z$



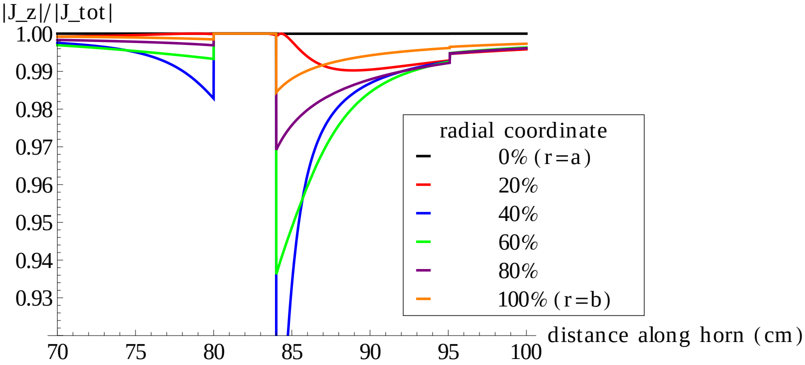
The radial current

I evaluate the radial current density at $r = (b + a)/2$:

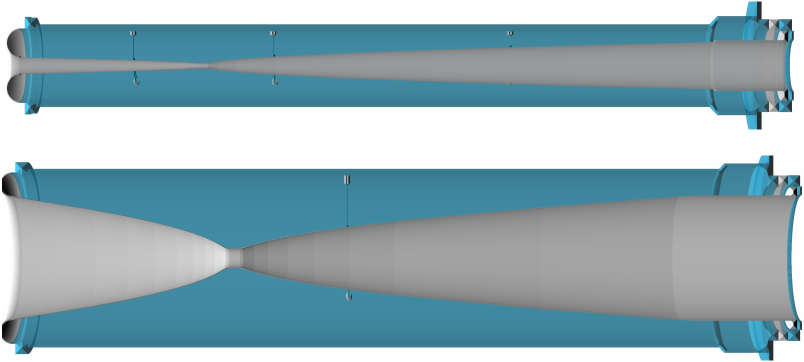


Fractional current density along z

Here is $J_z / J_{\text{total}} = 1/\sqrt{(J_r/J_z)^2 + 1}$:

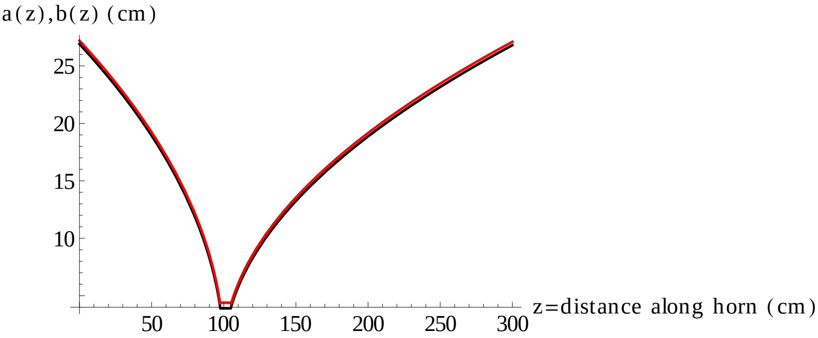


Horn 1 vs Horn 2

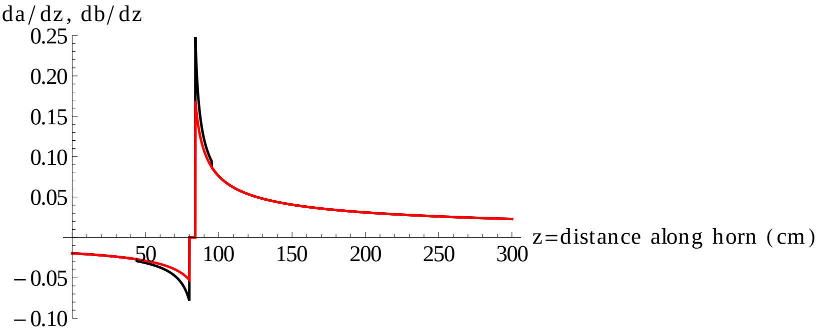


Horn 2 - $a(z)$ and $b(z)$

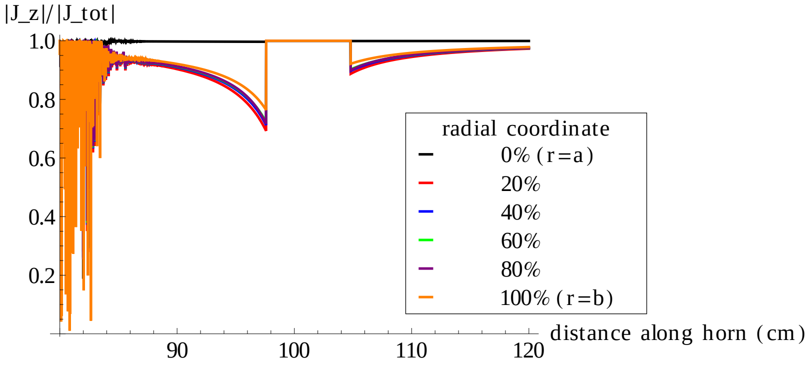
My best estimate of the shape from a piecewise function in a blurry old document:



Horn2 - $\partial a/\partial z$ and $\partial b/\partial z$

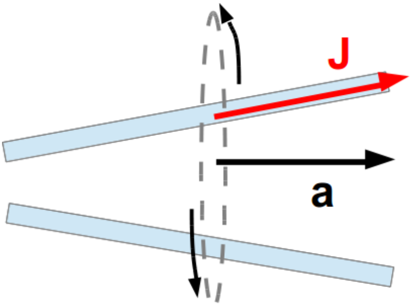


Fractional current density along z



How should we interpret J_z and J_r.

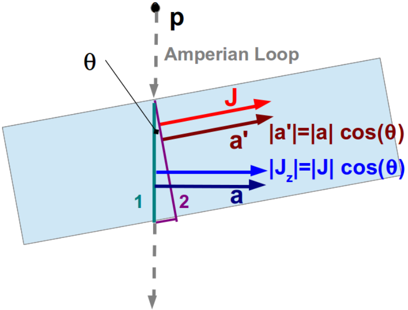
Consider a simple model for the horn inner conductor, with uniform current distribution:



Ampere's Law: $\oint \mathbf{H} \cdot \mathrm{d}\ell = \mu \int_{\text{surface}} \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \mu \int_{\text{surface}} J_z da$

How should we interpret J_z and J_r.

Of course, one can pick any surface in which to employ Ampere's Law.



Here, the current integrated through surface (2) must be equal to the input current in the horns (barring losses). And from that, at point p, you get $|H| = \mu I_{\text{in}}/2\pi r$. Thus $|J_z|$ must be such that you get the same answer. Not all of the current flows along z, but there is a larger area on surface (1) than on (2).

J_r implies E_r

- Ohm's law says $\mathbf{J} = \sigma \mathbf{E}$, so there is a radial electric field. Is this significant?
- The analytic expression for J_r is quite messy, but looking back at the semi-infinite plane case, we find that $E_y = 1/\sigma \partial H_x / \partial z = \frac{-k}{\sigma} H_0 e^{-kz} e^{-i\omega t}$
- Recall $k = \frac{i+1}{\delta}$
- The magnitude of E_y is just $\frac{H_0}{\sigma\delta}$.
- Take the largest value of H_0, which is at the horn neck, $r = 1.35\,\mathrm{cm}$ and $H_0 = 2.7\,\mathrm{T}$.
- $E_y = 2.33 \times 10^{-5}\,\mathrm{V/m}$ using $\rho = 1/\sigma = 5.1 \times 10^{-8}\,\Omega/\mathrm{m}$
- The magnitude of E_r in a cylindrical geometry cannot be very different.

Totally negligible, as expected since σ is large and δ not too small.

Outline

Conclusions

- Thanks to Bob Zwaska and Laura Fields for tipping me off about the disagreement with LBNE.
- Have reassesed the skin depth effect from first principles.
- I get a very different solution from what has been historically assumed.
- My solution is equivalent to a uniform distribution for $\delta > 5\,\mathrm{mm}$. This is good, since our CV is a uniform distribution.
- Seems we may be overestimating the systematic here.
- There is current in both the radial and longitudinal directions, but the radial current matters little and the longitudinal is being handled correctly.

The bigger picture

- MINOS beam fits have traditionally wanted to pull pretty hard on the horn current magnitude and/or distribution.
- Even so, after adjusting for the target not being right at $z = 10\,\mathrm{cm}$, we still have trouble getting the position and shape of the peak exactly right.
- MINERvA additionally employs hadron production data to reweight the flux and then tries beamfits. There appear to be similar data/MC discrepancies that are hard (but perhaps not impossible) to attribute to hadron production.
- The situation is troubling enough to spur the investigations reported in this talk.
- But, there doesn't really appear to be a current-distribution mismodeling. This is causing me to question some assumptions.
- My questions for "experts" are in the following slides. Some are hardware and some are software.

My questions for experts

- How well do we really know δ?
  - This includes the conductivity and its frequency and temperature dependence.
  - Also, around the time of the beam spill ($t = 0$), how sure are we that the dependence is $\cos(\omega t)$?
- How sure are we that we have the peak of the magnetic field in the horns timed in on the beam spill?
- How sure are we that the current going in on each stripline has the same ω and phase?
- What about the current in the horns? How recent is the calibration? Are we sure all the current flows in both horns?

My questions for experts

- Do we know the as-built horns agree with engineering drawings and our simulation? For example, could the neck be a bit narrower than we think?
- Do we actually put the field in the right place in the simulation? Are we sure the field doesn't extend outside the horns?
- Do we know that the field used to compute the Lorentz force is the field that our G4 magnetic field classes return?
- My derivation assumes boundary conditions on the field. Can we go the other direction with some kind of dynamic (time domain) calculation that inserts current into the horns?